

September 2026 - Problem of the Month

The Last Number | Complete Solution

Problem.

The board initially contains 1, 2, 3, ..., 2026.

In each move, erase two numbers a and b and write

$c = a + b + ab$ in their place.

Continue until a single number N remains.

Does N depend on the choices of a and b ?

Step 1. Look for a quantity that stays unchanged.

The key algebraic observation is

$$\mathbf{c + 1 = a + b + ab + 1 = (a + 1)(b + 1).}$$

This turns the unusual addition rule into multiplication.

Step 2. Define the invariant.

Suppose the current numbers are $x_1, x_2, \dots, x_k$.

Define $P = (x_1 + 1)(x_2 + 1) \dots (x_k + 1)$.

We claim that P is the same before and after every move.

Indeed, let R be the product of $(x + 1)$ over all numbers other than the selected a and b .

Before the move: $P_{\text{before}} = R(a + 1)(b + 1)$.

After the move: $P_{\text{after}} = R(c + 1)$.

Since $c + 1 = (a + 1)(b + 1)$, we obtain

$P_{\text{after}} = P_{\text{before}}$.

Therefore P is an invariant of the entire process.

The Last Number | Solution continued

Step 3. Evaluate the invariant at the start.

Initially the numbers are 1, 2, ..., 2026, so

$$\begin{aligned} P &= (1 + 1)(2 + 1) \dots (2026 + 1) \\ &= 2 \times 3 \times 4 \times \dots \times 2027 = 2027!. \end{aligned}$$

Step 4. Evaluate it at the end.

After 2025 moves, only one number N remains.

The same invariant is now simply $P = N + 1$.

Consequently, $N + 1 = 2027!$, and hence

FINAL ANSWER: $N = 2027! - 1$.

The final number does NOT depend on the choices.

Why this is a complete proof

Every move preserves P , regardless of which pair is chosen. Thus all possible sequences of moves lead to exactly the same value of N .

A small example (to see the idea)

Start with 1, 2, 3. Choose 1 and 2: $1 + 2 + 2 = 5$.

Then choose 5 and 3: $5 + 3 + 15 = 23$.

Alternatively: combine 2 and 3 to get 11, then combine 1 and 11 to get 23 again.

In both cases: $(1 + 1)(2 + 1)(3 + 1) = 24 = 23 + 1$.